Announcements

1. OH have been set

   M 4-5 PM
   T 2-3 PM
   W 4-5 PM
   R 3-4 PM
   F 10-11 AM

2. Mon. June 28th is a holiday

Ch. 2 First-Order DEs

We'll focus on DEs of the form

$$\frac{dy}{dt} = f(t,y)$$

2.1 Linear DEs and Method of Integrating Factors

Our most general first-order linear ODE is

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

in standard form. Here, p and g are given.

Sometimes we write

$$P(t)\frac{dy}{dt} + Q(t)\,y = G(t).$$

Using the the product rule, we can rewrite the DE so that we can directly integrate to get the solution.

Ex

$$\sin t \frac{dy}{dt} + \cos t \, y = t$$

$$\frac{d}{dt}(\sin t \, y) = t$$

$$\int \frac{d}{dt}(\sin t \, y)\, dt = \int t \, dt$$

$$\sin t \, y = \frac{t^2}{2} + C$$

$$y = \frac{\frac{t^2}{2} + C}{\sin t}$$

$$\frac{d}{dt}(\sin t \, y)$$

$$= \frac{d}{dt}(\sin t) \cdot y + \sin t \cdot \frac{d}{dt}(y)$$

$$= \cos t \cdot y + \sin t \, y'$$

We are not always so lucky and able to directly factor using the product rule. Multiplying the DE by an integrating factor $\mu(t)$ will allow us to rewrite the DE to integrate directly.

<u>Derivation of the integrating factor:</u>

We have the 1st order linear ODE

$$\frac{dy}{dt} + p(t)y = g(t)$$

which we will multiply by our to-be-determined integrating factor $\mu(t)$.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

We want the LHS to be written as

$$\frac{d}{dt}(r(t)y) = r(t)\frac{dy}{dt} + r'(t)y$$

So we see that we want

$$r(t) = \mu(t)$$

$$r'(t) = \mu(t)p(t)$$

therefore μ must satisfy

$$\mu'(t) = \mu(t)p(t)$$

Now,

$$\frac{\frac{d\mu}{dt}}{\mu(t)} = p(t)$$

$$\int \frac{d}{dt}\left(\ln \mu(t)\right) dt = \int p(t)\, dt$$

$$\ln \mu(t) = \int p(t)\, dt$$

$$\mu(t) = e^{\int p(t)\, dt}$$

Going back to solving the DE,

$$\int \frac{d}{dt}\left(\mu(t)\, y\right) dt = \int \mu(t)\, g(t)\, dt$$

$$\mu(t)\, y = \int \mu(t)\, g(t)\, dt$$

$$y = \frac{1}{\mu(t)} \int \mu(t)\, g(t)\, dt$$

or

$$y = e^{-\int p(t)\, dt} \int e^{\int p(t)\, dt}\, g(t)\, dt.$$

<u>Ex</u> Solve the IVP

$$(t^2+1)y' + y = t \quad , \quad y(0)=1$$

Converting to standard form, we have

$$y' + \underbrace{\frac{1}{t^2+1}}_{p(t)} y = \frac{t}{t^2+1}$$

Our integrating factor is

$$\mu(t) = e^{\int \frac{1}{t^2+1}\,dt}$$
$$= e^{\tan^{-1} t}$$

Multiply the DE by $\mu(t)$ and get

$$\underbrace{e^{\tan^{-1} t}\, y' + \frac{e^{\tan^{-1} t}}{t^2+1}\, y}_{} = \frac{t}{t^2+1}\, e^{\tan^{-1} t}$$

$$\frac{d}{dt}\left(e^{\tan^{-1} t}\, y\right) = \frac{t}{t^2+1}\, e^{\tan^{-1} t}$$

$$\frac{d}{dt}\left(e^{\tan^{-1}t}y\right) = \underline{e^{\tan^{-1}t}\left(\frac{1}{1+t^2}\right)y} + \underline{e^{\tan^{-1}t}\frac{dy}{dt}}$$

$$\int \frac{d}{dt}\left(e^{\tan^{-1}t}y\right) = \int \frac{t}{t^2+1}e^{\tan^{-1}t}\,dt$$

$$e^{\tan^{-1}t}y = \int_0^t \frac{s}{s^2+1}e^{\tan^{-1}s}\,ds + C$$

$$y = e^{-\tan^{-1}t}\int_0^t \frac{s}{s^2+1}e^{\tan^{-1}s}\,ds + Ce^{-\tan^{-1}t}$$

Our initial condition is $y(0) = 1$, so

$$1 = y(0) = 0 + C$$

$$C = 1$$

$$\Rightarrow \quad y = e^{-\tan^{-1}t}\int_0^t \frac{s}{s^2+1}e^{\tan^{-1}s}\,ds + e^{-\tan^{-1}t}$$

Remark We see that we are left with an integral in our solution. This may seem unsatisfactory, but we can use a computer to approximate the definite integral which yields the function y.

$\int_0^x x\,dx$

$\int_0^x s\,ds$
↑
"dummy variable"

$\frac{d}{dx}\int_0^x s\,ds = x$

$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$

§ 2.2 Separable DE

Consider our general 1st order ODE

$$\frac{dy}{dx} = f(x,y) \quad \text{with IC } y(x_0)=y_0$$

Suppose $f(x,y) = g(x)h(y)$. i.e. we can

Separate f into a product of a function depending only on x and a function depending only on y.

Then,

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$-g(x) + \frac{1}{h(y)}\frac{dy}{dx} = 0$$

Suppose $G'(x) = -g(x)$ and $H'(y) = \frac{1}{h(y)}$.

Then

$$G'(x) + H'(y)\frac{dy}{dx} = 0$$

and from the chain rule, we see that we have

$$\frac{d}{dx}(G(x) + H(y)) = G'(x) + H'(y)\frac{dy}{dx}$$

$$\frac{d}{dx}\left(G(x) + H(y) \right) = 0$$

so

$$G(x) + H(y) = C$$

Finally, the IC $y(x_0) = y_0$ gives

$$G(x_0) + H(y_0) = C$$

$H(y(x))$
$H(y(x_0))$
$H(y_0)$

and

$$G'(x) = -g(x)$$

$$\int_{x_0}^{x} G'(s)\,ds = \int_{x_0}^{x} -g(s)\,ds$$

$$G(x) - G(x_0) = \int_{x_0}^{x} -g(s)\,ds$$

$$H'(y) = \frac{1}{h(y)}$$

$$\int_{y_0}^{y} H'(s)\,ds = \int_{y_0}^{y} \frac{1}{h(s)}\,ds$$

$$H(y) - H(y_0) = \int_{y_0}^{y} \frac{1}{h(s)}\,ds$$

Thus,

$$G(x) - G(x_0) + H(y) - H(y_0) = 0$$

$$\int_{x_0}^{x} -g(s)\,ds + \int_{y_0}^{y} \frac{1}{h(s)}\,ds = 0$$

gives us an implicit representation of the solution. The final step is to solve for y as a function of x.

<u>Note</u>: Often, determining an explicit formula is impossible, but you can use numerical methods to find approximate values of y for given values of x.

<u>Ex</u> Solve the following IVP

$$\frac{dy}{dx} = \frac{4x-x^3}{4+y^3} \qquad y(0) = 1$$

We see that the DE is separable since

$$\frac{4x-x^3}{4+y^3} = f(x)\,h(y) \quad \text{where} \quad f(x) = 4x-x^3$$

$$h(y) = \frac{1}{4+y^3}$$

Now,

$$(4+y^3)\frac{dy}{dx} = 4x-x^3$$

$$\int \frac{d}{dx}\int_1^y 4+s^3\,ds\,dx = \int 4x-x^3\;dx$$

$$\int_1^y 4+s^3\,ds = \frac{4x^2}{2} - \frac{x^4}{4} + C$$

$$4s+\frac{s^4}{4}\Big|_1^y = 2x^2 - \frac{x^4}{4} + C$$

$$4y+\frac{y^4}{4} - \left(4+\frac{1}{4}\right) = 2x^2 - \frac{x^4}{4} + C$$

$$16y + y^4 - 4\left(\frac{17}{4}\right) = 8x^2 - x^4 + C$$

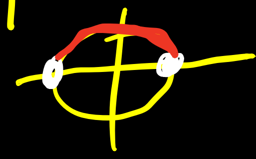
Now, $y(0)=1$, so

$$16 + 1 - 17 = C$$

$$\Rightarrow C = 0$$

Implicit Eqn

$x^2+y^2=1$



Thus,

$$16y + y^4 = 8x^2 - x^4 + 17$$

For what values of x is this solution valid?

The implicit solution makes sense for all values of x, but the DE blows up when $4+y^3=0$ or $y=(-4)^{1/3}$. The corresponding values of x are solutions to

$$16(-4)^{\frac{1}{3}} + (-4)^{\frac{4}{3}} = 8x^2 - x^4 + 17$$

and so $x \approx \pm 3.35$. Therefore, the domain of validity is

$$x \in [-3.35, 3.35].$$

2.3 Modeling with 1st-order DEs

1. Construct the model
2. Analysis of the model
3. Comparison with Experiment or Observation